Determination of Cycling Speed Using a Closed-form Solution from Nonlinear Dynamic Equations

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Abstract

The theme of this paper is to develop a closed-form method to solve nonlinear equations for cycling speed. The nonlinear equations are derived from force and power balance. The closed-form solution for force equation is straightforward, yet the closed-form solution for power equation is obtained by means of partitioning technique. For simulation purpose, the effect of air drag coefficient, frontal area, rolling resistance, road grade, and power consumption is studied. With the closed-form solution, the evaluation of cycling speed becomes easy.

Introduction

The estimation of vehicle speed has been of interest to researchers (Gillespie, 1992; Wong, 1993; Lieh, 1995). Their studies were concentrated on automotives. A number of researchers investigated the effect of aerodynamics on cycling speed (Whitt and Wilson, 1982; Gross et al., 1983; Kyle and Zahradnik, 1987; Brandt, 1988; Kyle, 1988). However, there is lack of a general closed-form solution for cycling speed; therefore, the estimation of vehicle speed was normally done in a tedious and time-consuming spreadsheet or in a commercial program, such as ADAMS, DADS-3D or VEHSIM. Lieh (2002) indicated that a closed-form method for determining vehicle speed could simply the calculation procedure.

This paper will adopt such a closed-form method. The nonlinear equations for cycling are formulated and expressed in terms of velocity and key vehicle parameters. The integration of these equations is conducted symbolically and is used to compute the maximum value and time history of speed for various cases.

Traction and Resistant Forces

As shown in Figure 1, the major external forces to be overcome by the tire traction force (F_T) during cycling are

F_a, aerodynamic force
F_r, rolling resistance ($F_{rf} + F_{rr}$)
F_i, inertia force
F_g, gravitational force

The dynamic equilibrium of the system along the longitudinal direction can be written as

$$F_T = F_a + F_r + F_i + F_g \tag{1}$$

Sum the moments about point ***A*** (without considering the airlift effect), the normal load on the driving wheel is

$$W_r = \frac{WL_a\cos(\theta) + F_a h_a + mh\dot{v} + Wh\sin(\theta)}{L} \tag{2}$$

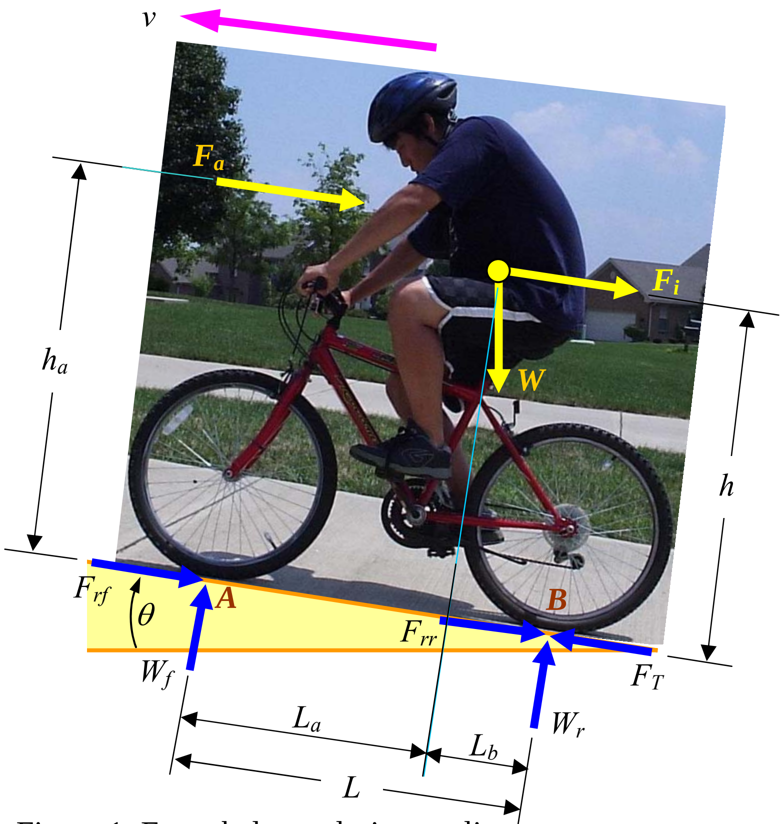

Figure 1: Force balance during cycling.

Define μ_p the peak coefficient of tire/road friction, N the normal load on the driving wheel, C_i the tire longitudinal stiffness, and S the slip. The tire traction force may be expressed as follows (Wong, 1993):

$$F_T = \mu_p N\left(1 - \frac{\mu_p N}{4SC_i}\right) \tag{3}$$

For the current study, it is assumed that the traction force is near its peak value such that the maximum theoretical speed may be obtained, i.e.

$$F_T = \mu_p N \tag{4}$$

Define m the mass of bicycle plus the rider, ρ the air density, C_d the air drag coefficient, A_f the frontal area, v the forward velocity, θ the grade angle, g the gravity, and f_o and f_1 the rolling resistance coefficients (f_1 is usually very small but is included here for generality). The air drag, rolling resistance, inertia and gravitational forces can be expressed in the following form (Gillespie, 1992; Wong, 1993):

$$F_a = \frac{\rho}{2} C_d A_f v^2 \tag{5}$$

$$F_r = (f_o + f_1 v^2)W \tag{6}$$

$$F_i = m\dot{v} \tag{7}$$

$$F_g = W \sin(\theta) \tag{8}$$

Where, $\dot{v} = \dfrac{dv}{dt}$ and $W = mg$. Substituting these forces into Eqn (1), a nonlinear equation describing the dynamic equilibrium can be written in the following form

$$s_1\dot{v} = s_2 - s_3 v^2 \tag{9}$$

Where, the expressions for s_i are

$$s_1 = m\left(1 - \mu_p \frac{h}{L}\right) \tag{10a}$$

$$s_2 = W\left\{\mu_p\left[\frac{L_a}{L}cos(\theta) + \frac{h}{L}sin(\theta)\right] - f_o - sin(\theta)\right\} \tag{10b}$$

$$s_3 = \frac{\rho}{2}\left(1 - \mu_p \frac{h_a}{L}\right)C_d A_f + f_1 W \tag{10c}$$

Eqn (9) represents the tire traction capability of a bicycle when its power availability is adequate or unlimited. Since the bicycle is a rear-wheel drive system, the rear normal load stated in Eqn (2) is used to determine the traction force, i.e. $N = W_r$. The theoretical maximum speed is found by setting $\dot{v} = 0$, as a result, Eqn (9) is reduced to

$$s_2 - s_3 v^2 = 0 \tag{11}$$

The maximum speed based on tire traction capability is

$$\begin{aligned} v_{Tm} &= \sqrt{\frac{s_2}{s_3}} \\ &= \sqrt{\frac{W\left\{\mu_p\left[\frac{L_a}{L}cos(\theta) + \frac{h}{L}sin(\theta)\right] - f_o - sin(\theta)\right\}}{\frac{\rho}{2}\left(1 - \mu_p \frac{h_a}{L}\right)C_d A_f + f_1 W}} \end{aligned} \tag{12}$$

To find the time history of velocity, it requires an integration of Eqn (9). To facilitate the process, the equation is re-arranged as

$$dt = s_1 \frac{dv}{s_2 - s_3 v^2} \tag{13}$$

The symbolic or closed-form integral of Eqn (13) is

$$v = \frac{\sqrt{s_2}}{\sqrt{s_3}} tanh\left[\frac{\sqrt{s_2 s_3}}{s_1}(t - t_o) + tanh^{-1}\left(\frac{\sqrt{s_3}v_o}{\sqrt{s_2}}\right)\right] \tag{14a}$$

$$= v_{Tm}\, tanh\left[\frac{s_3}{s_1}v_{Tm}(t - t_o) + tanh^{-1}\left(\frac{v_o}{v_{Tm}}\right)\right] \tag{14b}$$

Where v_o is the initial velocity at time t_o. If the bicycle starts from rest, i.e., $v_o = 0$ at $t_o = 0$, the velocity can be simplified to

$$v = \sqrt{\frac{s_2}{s_3}} tanh\left(\frac{\sqrt{s_2 s_3}}{s_1} t\right) \tag{15a}$$

$$= v_{Tm}\, tanh\left(\frac{s_3}{s_1} v_{Tm}\, t\right) \tag{15b}$$

Power Equation

The above closed-form solution for maximum speed represents the maximum tire traction capability assuming that an adequate or unlimited power can be delivered by the rider. However rider's output is constrained because of human's physical strength. This means Eqn (12) can only be used for motorcycle case. To obtain the maximum speed for a human powered vehicle, the capability of rider's power output has to be considered.

Define P_T the rider's total available power. The power transmitted to the rear wheel is modified by the cycling efficiency η

$$\begin{aligned} P_w &= \eta P_T \\ &= \left(F_a + F_r + F_i + F_g\right) v \end{aligned} \tag{16}$$

Substituting Eqns (5-8) into Eqn (16) results in a nonlinear differential equation in the following form

$$\dot{v} = \frac{-1}{mv}\left(r_1 v^3 + r_2 v - r_3\right) \tag{17}$$

To facilitate integration, this equation is written as

$$dt = -m\left(\frac{v dv}{r_1 v^3 + r_2 v - r_3}\right) \tag{18}$$

Where the constants are given below

$$r_1 = \frac{1}{2}\rho C_d A_f + f_1 W \tag{19a}$$

$$r_2 = W[f_o + sin(\theta)] \tag{19b}$$

$$r_3 = \eta P_T \tag{19c}$$

It can be seen from Eqn (17) when the bicycle starts from rest, i.e. $v = 0$, the equation is singular. This implies that the acceleration can be high when the speed is near zero. Since the equation is very nonlinear, a simple way to integrate is to partition Eqn (18) into

$$\frac{-r_1}{m} dt = \frac{a_1}{v + b_1} dv + \frac{a_2 v + a_3}{v^2 + b_2 v + b_3} dv \tag{20}$$

Where a_1, a_2, a_3, b_1, b_2, b_3 are functions of r_1, r_2, and r_3, and they are determined by the following sequential steps:

Step-1: Calculate b_2 from the following equation

$$b_2 = \frac{1}{6r_1}\sqrt[3]{\left(108r_3 + 12\sqrt{\frac{3(4r_2^3 + 27r_3^2 r_1)}{r_1}}\right)r_1^2} - \frac{2r_2}{\sqrt[3]{\left(108r_3 + 12\sqrt{\frac{3(4r_2^3 + 27r_3^2 r_1)}{r_1}}\right)r_1^2}} \tag{21}$$

Step-2: Calculate b_1 and b_3 with

$$b_1 = -b_2$$

$$b_3 = b_2^2 + \frac{r_2}{r_1}$$

Step-3: Calculate a_1, a_2, a_3 with

$$a_1 = \frac{b_2}{3b_2^2 + \frac{r_2}{r_1}}$$

$$a_2 = -a_1$$

$$a_3 = 2a_2 b_2 + 1$$

The symbolic solution or closed-form integral (Dwight, 1985) of Eqn (20) is found to be

$$\begin{aligned} t = t_o + \frac{m}{r_1}\Bigg[& \left(\frac{a_2 b_2}{2} - a_3\right)\frac{2}{\sqrt{4b_3 - b_2^2}} tan^{-1}\left(\frac{2v + b_2}{\sqrt{4b_3 - b_2^2}}\right) - a_1 LN\left|v + b_1\right| \\ & - \frac{a_2}{2} LN\left|v^2 + b_2 v + b_3\right| - \left(\frac{a_2 b_2}{2} - a_3\right)\frac{2}{\sqrt{4b_3 - b_2^2}} tan^{-1}\left(\frac{2v_o + b_2}{\sqrt{4b_3 - b_2^2}}\right) \\ & + a_1 LN\left|v_o + b_1\right| + \frac{a_2}{2} LN\left|v_o^2 + b_2 v_o + b_3\right| \Bigg] \end{aligned} \tag{22}$$

From Eqn (18), the maximum speed is obtained by setting $\dot{v} = 0$. This leads to

$$v^3 + \frac{r_2}{r_1} v - \frac{r_3}{r_1} = 0 \tag{23}$$

The maximum speed based on a given power output is the real root of Eqn (23), which has the same solution as b_2, i.e.

$$v_{pm} = \frac{1}{6r_1}\sqrt[3]{\left(108r_3 + 12\sqrt{\frac{3(4r_2^3 + 27r_3^2 r_1)}{r_1}}\right) r_1^2} - \frac{2r_2}{\sqrt[3]{\left(108r_3 + 12\sqrt{\frac{3(4r_2^3 + 27r_3^2 r_1)}{r_1}}\right) r_1^2}} \tag{24}$$

Simulation Results

The simulation is conducted by varying air drag coefficient, frontal area, rolling resistance coefficient, mass, and road grade.

Figure 2 shows the maximum speed versus power for various major parameters: (a) is for the effect of air drag coefficient C_d, (b) is for the effect of frontal area A_f, (c) is for the effect of mass m, and (d) is for the effect of rolling resistance coefficient f_o.

Figure 3 shows the maximum speed versus power for various road grades and vehicle parameters. It is observed that if the air drag coefficient, front area, and rolling resistance coefficient are low (part (d): $C_d=0.2$, $A_f = 0.2m^2$, $f_o = 0.007$), a *60-kg* racer can reach *90 km/hr* on a flat road and *30 km/hr* on a *9%*-grade road if he can deliver *500* watts of power. However, if the air drag coefficient, front area, and rolling resistance coefficient increase, for example $C_d=0.35$, $A_f = 0.35m^2$, $f_o = 0.013$ in Part (a), the maximum speed can only reach *58 km/hr* on a flat road and *18 km/hr* on a *9%*-grade road.

For computing the time history of velocity, it requires to use Eqn (14) at the beginning then switch to Eqn (22) when the power of cycling is balanced. Figure 4 shows the velocity versus time for the case of $C_d=0.3$, $A_f = 0.3m^2$, $f_o = 0.011$, $m = 60\ kg$. It can be seen that the velocity can reach *42 km/hr* in *30* seconds if the rider can deliver *250* watts on a flat road (or *60 km/hr* for *500* watts). However, for the same time period and the same power output, the velocity can only reach *15 km/ hr* on a *9%*-grade road (or *28 km/hr*).

Summary

Based on the nonlinear equations derived from force and power equilibrium, closed-form solutions for determining cycling speeds are obtained. With the closed-form solutions, the estimation of maximum speeds becomes straightforward. And the evaluation of vehicle performance under the influence of vehicle parameters (such as air drag, frontal area, power, mass, and rolling resistance) becomes simple and easy.

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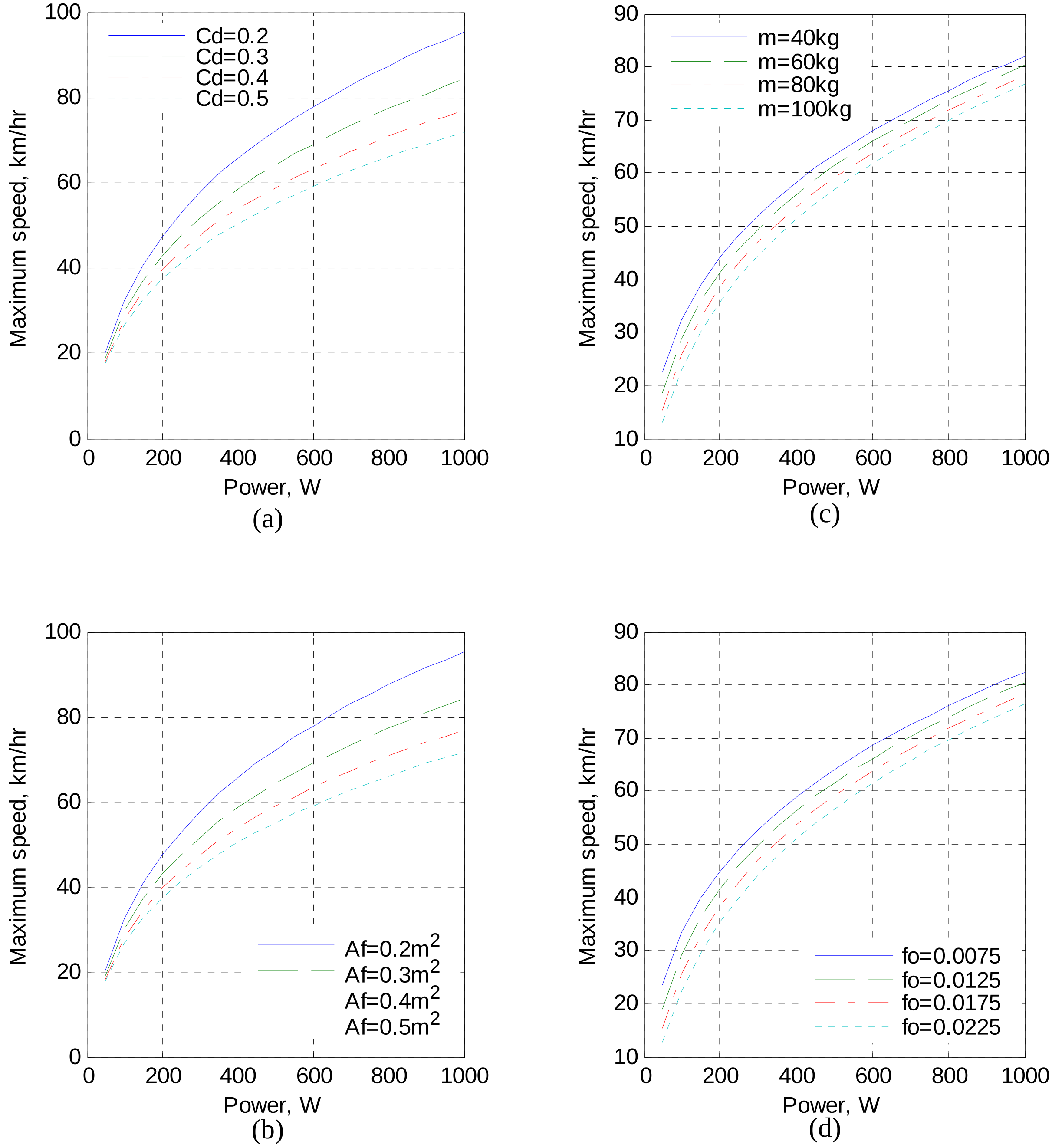

Figure 2: Maximum speed (v_{pm}) vs. power for various vehicle parameters, on a flat road ride.

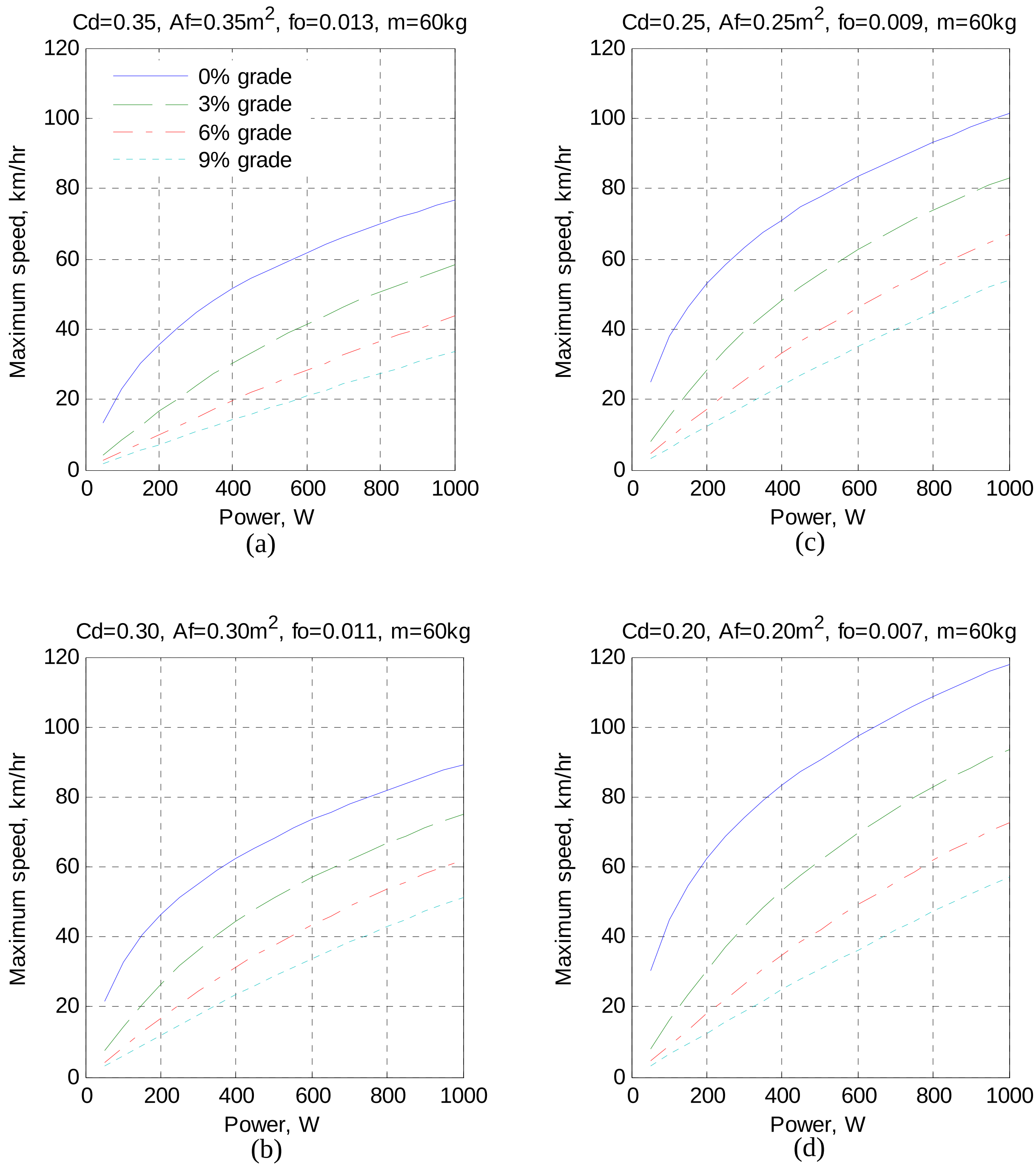

Figure 3: Maximum speed (v_{pm}) vs. power for various road grades and cycling parameters.

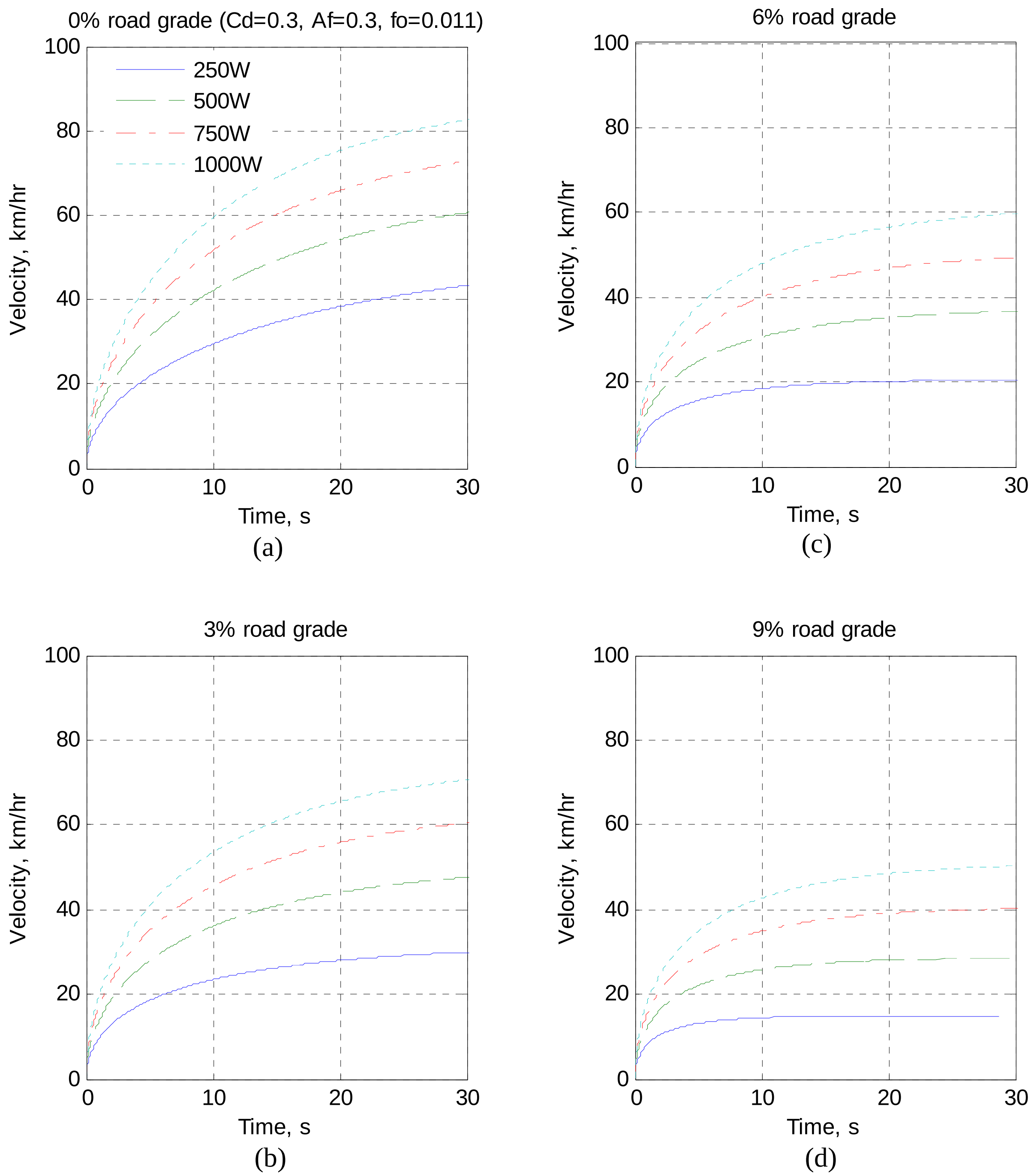

Figure 4: Velocity vs. time for various road grades (given: C_d=0.3, A_f = 0. 3m^2, f_o = 0.011, m = 60 kg).